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For $n \geq 5$ There is no Nontrivial Z_2-Measure on $L(R^n)$

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We prove the statement in the title. As a consequence, we note that there is no nontrivial Z_2-measure on $L(H)$, $\dim H = \infty$.

KEY WORDS: quantum logic; projection lattice; group-valued measure.

1. INTRODUCTION

In (Foulis, 2000), D. Foulis initiates and investigates a new approach to the theoretical foundation of quantum mechanics. The approach is based on the concept of *universal group* associated with group-valued measures on a quantum logic. Naturally, a better understanding of group-valued measures has become desirable (for some previous results, see (Greechie, 1971; Navara, 1994; Ovchinnikov, 1999; Pták, 1998; Weber, 1994), etc.). For the group Z_2 (the group $\{0, 1\}$ mod 2), however, the interest coming from the motivation indicated above combines with the interest in the intriguing question of "hidden variables" (see (Bell, 1966; Kochen and Specker, 1967; Mermin, 1993; Pitowsky, 1998; Svozil and Tkadlec, 1996), etc.). This fact, together with some initial observations on group-valued measures, has been communicated to us by J. Harding (Harding). Responding to this, we show that for the projection logic $L(R^n)$, where $n \geq 5$, there is no nontrivial Z_2-valued measure (alias "there is no generalized hidden variable on $L(R^n)$, $n \geq 5$").

2. NOTIONS AND RESULTS

We shall use the notion of *quantum logic* $(L, \leq, ')$ in its standard meaning (see e.g., (Pták and Pulmannová, 1991)), we do not require that $(L, \leq, ')$ be σ-complete. We harmlessly abuse the notation by referring to L only.

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The principal notion we want to study here is introduced in the following definition. (Recall that by Z_2, we shall denote the group {0, 1} with the "mod 2" addition.)

Definition 2.1. Let L be a quantum logic. A mapping s: $L \to Z_2$ is said to be a Z_2-measure if $s(0) = 0$ and $s(a \vee b) = s(a) + s(b)$ whenever $a \leq b'$.

In this note, we take up the fundamental case of L being the projection logic $L(R^n)$ of $R^n (n \in N)$. Obviously, $L(R^n)$ always possesses Z_2-measures – it suffices to set $s(a) = 0$ for each atom a of $L(R^n)$. Also, if we set $s(a) = 1$ for each atom, we obtain a Z_2-measure on $L(R^n)$, too. The situation becomes more interesting if we exclude these trivial cases. Let us say that a Z_2-measure s is *nontrivial* if there are atoms $a, b \in L(R^n)$ with $s(a) = 0$ and $s(b) = 1$. Clearly, $L(R^1)$ is too primitive to be checked and $L(R^2)$ possesses nontrivial measures. The case of $L(R^3)$ seems entirely open. (Obviously, beginning with $n = 3$, there is no standard two-valued measure on $L(R^n)$, see e.g., (Kochen and Specker, 1967).) For $L(R^4)$, we have a partial result (Theorem 2) which suggests that there is no nontrivial Z_2-measure on $L(R^4)$. For $L(R^n)$, $n \geq 5$, we show (Theorem 3) that there is no nontrivial Z_2-measure on this logic.

Theorem 2.1. *There is no nontrivial Z_2-measure, s, on $L(R^4)$ which satisfies $s(1) = 1$ (here 1 in the parentheses obviously means the identity projection on R^4).*

Proof: Take an orthogonal basis $B = \{e_1, e_2, e_3, e_4\}$ of R^4. For typographical reasons, let us adopt the convention that $\bar{1}$ denotes -1. Consider the collection of 36 vectors expressed in Table I in terms of their coordinates with respect to B. Observe that each column represents an orthogonal basis of R^4 and that each vector *occurs twice* in the collection. (This collection is a modification of that dealt with in (Peres, 1995).) Suppose that s: $L(R^n) \to Z_2$ is a nontrivial measure with $s(1) = 1$. Then if we restrict s to the one-dimensional subspaces generated by vectors belonging to a given column, we immediately see that the value 1 occurs either one or three times. Since we have nine columns, the total sum (mod 2) of all values the Z_2-measure s attained on all subspaces (vectors) of the collection is 1. But each vector occurs twice. This implies that the total sum must be 0 – a contradiction. □

Table I. Nine orthogonal bases of R^4

1000	1000	0100	1111	1111	$111\bar{1}$	$11\bar{1}\bar{1}$	$111\bar{1}$	$11\bar{1}1$
0100	0010	0010	$11\bar{1}\bar{1}$	$1\bar{1}1\bar{1}$	$11\bar{1}1$	$1\bar{1}1\bar{1}$	$1\bar{1}11$	$1\bar{1}11$
0011	0101	1001	$1\bar{1}00$	$10\bar{1}0$	$1\bar{1}00$	1001	$10\bar{1}0$	$100\bar{1}$
$001\bar{1}$	$010\bar{1}$	$100\bar{1}$	$001\bar{1}$	$010\bar{1}$	0011	0110	0101	0110

Theorem 2.2. *If $n \geq 5$, then there is no nontrivial Z_2-measure on $L(R^n)$.*

Proof: We provide the proof for $n = 5$ – the general case follows easily. Let s: $L(R^5) \to Z_2$ be a nontrivial Z_2-measure. Then there is an atom, $a \in L(R^5)$, such that $s(a) \neq s(1)$. It follows that $s(a^\perp) = 1$. But $a^\perp$ is a four-dimensional space and we could easily be able to construct a nontrivial Z_2-measure, t, on $L(R^4)$ with $t(1) = 1$. But this cannot be done in view of Theorem 2. The proof is complete (the extension to $L(R^n)$, $n > 5$, is straightforward). □

Observe that since $L(R^5)$ can be viewed as a quantum sublogic of $L(H)$ for an infinite-dimensional Hilbert space H (see e.g., (Hamhalter and Pták, 1992)), the result continues to hold true for the logics $L(H)$, dim $H = \infty$.

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